OCR
Oxford Cambridge and RSA

Friday 17 June 2016 – Afternoon

A2 GCE MATHEMATICS

4727/01 Further Pure Mathematics 3

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4727/01
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found inside the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. If additional space is required, you should use the lined page(s) at the end of the Printed Answer Book. The question number(s) must be clearly shown.
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER/INVIGILATOR

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Answer **all** the questions.

1 In this question, give all non-real numbers in the form $re^{i\theta}$ where $r > 0$ and $0 < \theta < 2\pi$.

(i) Solve $z^5 = 1$. **[2]**

(ii) Hence, or otherwise, solve $z^5 + 32 = 0$. Sketch an Argand diagram showing the roots. **[4]**

2 Find the shortest distance between the lines $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$. **[4]**

3 The differential equation

$$\frac{2}{y} - \frac{x}{y^2}\frac{dy}{dx} = \frac{1}{x^2}$$

is to be solved subject to the condition $y = 1$ when $x = 1$.

(i) Show that $y = \frac{1}{u}$ transforms the differential equation into

$$x\frac{du}{dx} + 2u = \frac{1}{x^2}.$$ **[3]**

(ii) Find y in terms of x. **[7]**

4 Let A be the set of non-zero integers.

(i) Show that A does not form a group under multiplication. **[2]**

(ii) State the largest subset of A which forms a group under multiplication. Show that this is a group. **[3]**

5 Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 85\cos x.$$ **[8]**

6 The planes Π_1 and Π_2 have equations

$$\mathbf{r}.\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 3 \quad \text{and} \quad \mathbf{r}.\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 5$$

respectively. They intersect in the line l.

(i) Find cartesian equations of l. **[4]**

The plane Π_3 has equation $\mathbf{r}.\begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} = 1$.

(ii) Show that Π_3 is parallel to l but does not contain it. **[3]**

(iii) Verify that $(2, 0, 1)$ lies on planes Π_1 and Π_3. Hence write down a vector equation of the line of intersection of these planes. **[3]**

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7 **(i)** Use de Moivre's theorem to show that

$$\sin 6\theta \equiv \cos\theta(6\sin\theta - 32\sin^3\theta + 32\sin^5\theta).$$ **[5]**

(ii) Hence show that, for $\sin 2\theta \neq 0$,

$$-1 \leqslant \frac{\sin 6\theta}{\sin 2\theta} < 3.$$ **[7]**

8 A non-commutative multiplicative group G of order eight has the elements

$$\{e, a, a^2, a^3, b, ab, a^2b, a^3b\},$$

where e is the identity and $a^4 = b^2 = e$.

(i) Show that $ba \neq a^n$ for any integer n. **[2]**

(ii) Prove, by contradiction, that $ba \neq a^2b$ and also that $ba \neq ab$. Deduce that $ba = a^3b$. **[6]**

(iii) Prove that $ba^2 = a^2b$. **[2]**

(iv) Construct group tables for the three subgroups of G of order four. **[7]**

END OF QUESTION PAPER

Question			Answer	Marks	Guidance	
1	(i)		$z=1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}$	M1 A1 [2]	$e^{2\pi i/5}$ soi	
	(ii)		$z^5=-32$ has a root -2, so roots are $-2, -2e^{2\pi i/5}, -2e^{4\pi i/5}, -2e^{6\pi i/5}, -2e^{8\pi i/5}$ Roots $-2, 2e^{7\pi i/5}, 2e^{9\pi i/5}, 2e^{\pi i/5}, 2e^{3\pi i/5}$ Argand diagram	M1 A1 M1 A1 [4]	Use part **(i)** or from scratch cao with $r>0, 0<\theta<2\pi$ (allow $2e^{\pi i}$ for -2) one root in each quadrant plus one on real axis axes and roots labelled. Roots equal moduli and equiangular spacing	
2			$\begin{pmatrix}1\\2\\-1\end{pmatrix}\times\begin{pmatrix}3\\0\\1\end{pmatrix}=\begin{pmatrix}2\\-4\\-6\end{pmatrix}=-2\begin{pmatrix}-1\\2\\3\end{pmatrix}$ $\begin{pmatrix}2\\1\\0\end{pmatrix}-\begin{pmatrix}-1\\1\\2\end{pmatrix}=\begin{pmatrix}3\\0\\-2\end{pmatrix}$ Shortest distance $=\frac{\left\lvert\begin{pmatrix}3\\0\\-2\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\3\end{pmatrix}\right\rvert}{\sqrt{1^2+2^2+3^2}}$ $=\frac{9}{\sqrt{14}}$ or 2.41	M1 A1 M1 A1 [4]	at least 2 correct values for the cross product or method shown Any multiple	
3	(i)		$\frac{dy}{dx}=-u^{-2}\frac{du}{dx}$ $2u-xu^2\left(-u^{-2}\frac{du}{dx}\right)=\frac{1}{x^2}$ $x\frac{du}{dx}+2u=\frac{1}{x^2}$	M1 M1 A1 [3]	Differentiate Substitute **ag** Convincingly shown	

Question			Answer	Marks	Guidance	
	(ii)		$\frac{du}{dx}+\frac{2u}{x}=\frac{1}{x^3}$			
			$I=\exp\left(\int \frac{2}{x}dx\right)=e^{2\ln x}$	M1	$e^{k\ln x}$	
			$=x^2$	A1		incorrect IF means no further marks can be gained
			$\frac{d}{dx}(x^2u)=x^{-1}$	M1	for LHS, multiply and recognise derivative	if RHS is not multiplied by IF then no further marks can be gained
			$x^2u=\ln x+A$	A1		or $\ldots = lnkx$
			$u=(\ln x+A)/x^2$			
			$y=x^2/(\ln x+A)$	M1	for y = reciprocal of 'their u'	
			$x=1, y=1 \Rightarrow 1=\frac{1}{0+A} \Rightarrow A=1$	M1		or $k=e$
			$y=\frac{x^2}{\ln x+1}$	A1 [7]	oe without fractions within fractions	or $y=\frac{x^2}{\ln ex}$
4	(i)		$\forall n, 1n=n1=n$ so 1 is identity	M1	Identify identity	can be implicit for M1
			But not all integers have an inverse, e.g. $2n\neq 1$ for any n	A1 [2]	Complete argument (example or general)	
	(ii)		$\{-1,1\}$	B1*		
			Demonstrates closure, references associativity references identity		...without contradiction	
			$(-1)^{-1}=-1$ (and $1^{-1}=1$) so inverses	*B2 [3]	B1 for any two of these Dep on 1st B1	

Question			Answer	Marks	Guidance	
5			AE: $\lambda^2 + 2\lambda + 10 = 0$ $\lambda = -1 \pm 3i$ CF: $e^{-x}(A\cos 3x + B\sin 3x)$	B1 B1 B1ft	condone $Ae^{(-1+3i)x} + Be^{(-1-3i)x}$	ft on complex λ only
			PI: $y = a\cos x + b\sin x$	B1		trial function $y = a\cos x$ scores max of B0 M1 M0 at this stage
			$y' = -a\sin x + b\cos x$ $y'' = -a\cos x - b\sin x$ In DE: $-a\cos x - b\sin x + 2(-a\sin x + b\cos x) + 10(a\cos x + b\sin x) = 85\cos x$	M1*	Differentiate twice and substitute	
			$-a + 2b + 10a = 85$ $-b - 2a + 10b = 0$	M1*	Compare coefficients	
			$a = 9, b = 2$	A1	PI correct	
			GS: $y = 9\cos x + 2\sin x$ $+e^{-x}(A\cos 3x + B\sin 3x)$	*A1ft [8]	Their CF (of standard form) + their PI	dep on gaining both M1 marks
6	(i)		$\begin{pmatrix}1\\2\\1\end{pmatrix} \times \begin{pmatrix}2\\1\\4\end{pmatrix} = \begin{pmatrix}7\\-2\\-3\end{pmatrix}$	M1A1		
			finds point on both planes	B1	e.g. (0,1,1)	or $\left(\frac{7}{3}, \frac{1}{3}, 0\right)$ or $\left(\frac{7}{2}, 0, -\frac{1}{2}\right)$
			$\frac{x}{-7} = \frac{y-1}{2} = \frac{z-1}{3}$	A1 [4]	oe	
	ALT		$x + 2y + z = 3$ $2x + y + 4z = 5$			
			$3x + 7z = 7$	M1 A1	Attempts to find at least 1 equation 2 correct equations	
			$2x + 7y = 7$			or $3y - 2z = 1$
			$\frac{x}{-7} = \frac{y-1}{2} = \frac{z-1}{3}$	M1A1 [4]	oe of the form f(x) = g(y)= h(z)	

Question			Answer	Marks	Guidance	
	(ii)		$\begin{pmatrix}-7\\2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\5\\-1\end{pmatrix}=-7+10-3=0$	M1	For scalar product, either shows method or gives answer of zero	
			$\Rightarrow l \parallel \Pi_3$	A1	for A1 must have working out line for scalar product	
			$(0,1,1)$ is on line, but $\begin{pmatrix}0\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\5\\-1\end{pmatrix}=4\neq 1$ so not on plane	B1 [3]		
	ALT		$x+5y-z=1$ $7\lambda+5(1-2\lambda)-(1-3\lambda)=1$	M1A1		
			$\Rightarrow 4=1$ inconsistent, so l is parallel and not on plane	A1 [3]		
	(iii)		$2+2\times 0+1=3$ (so on Π_1)			must show working for at least one plane
			$\begin{pmatrix}2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\5\\-1\end{pmatrix}=1$ (so on Π_3)	B1	Verify both	
			Line has equation $\mathbf{r}=\begin{pmatrix}2\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}-7\\2\\3\end{pmatrix}$	M1 A1 [3]	oe vector form in cartesian form M1 only	if cross product calculated incorrectly then M0A0
7	(i)		$\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$	B1	Use de Moivre	or $\sin 6\theta = Im(\cos\theta + i\sin\theta)^6$
			$=\cos^6\theta + 6i\sin\theta\cos^5\theta - 15\sin^2\theta\cos^4\theta - 20i\sin^3\theta\cos^3\theta + 15\sin^4\theta\cos^2\theta + 6i\sin^5\theta\cos\theta - \sin^6\theta$	B1	All terms correct	
			$\sin 6\theta = 6\sin\theta\cos^5\theta - 20\sin^3\theta\cos^3\theta + 6\sin^5\theta\cos\theta$	M1	Compare imaginary parts	
			$=\cos\theta\,(6\sin\theta\,(1-\sin^2\theta)^2 - 20\sin^3\theta\,(1-\sin^2\theta) + 6\sin^5\theta)$	M1	Take out factor of $\cos\theta$ and give other factor in terms of $\sin\theta$only	
			$=\cos\theta\,(32\sin^5\theta - 32\sin^3\theta + 6\sin\theta)$	A1 [5]	**ag** Convincingly shown, having been explicit about taking imaginary parts	must have $\sin 6\theta = \cdots$'final line'

Question			Answer	Marks	Guidance	
	(ii)		$\frac{\sin 6\theta}{\sin 2\theta} = \frac{\cos\theta\,(32\sin^5\theta - 32\sin^3\theta + 6\sin\theta)}{2 sin\theta cos\theta}$ $= 16\sin^4\theta - 16\sin^2\theta + 3$	M1		
			$= 4(2\sin^2\theta - 1)^2 - 1$	M1	Complete the square	
				A1		**M1A1** for showing stationary points occur when $sin^2\theta = 0{,}1\ or\ \frac{1}{2}$
			$\therefore \frac{\sin 6\theta}{\sin 2\theta} \geq -1$	M1	deduces lower bound	if using calculus, must convince for nature of stationary points for each M1 here
			$0 \leq 2\sin^2\theta \leq 2$ $\therefore (2\sin^2\theta - 1)^2 \leq 1$	M1	deduces upper bound	can omit line 1or 2 from the workings here, but not for final A mark
			$\therefore 4(2\sin^2\theta - 1)^2 - 1 \leq 3$ $\Rightarrow -1 \leq \frac{\sin 6\theta}{\sin 2\theta} \leq 3$		**SC** if none of marks 2 to 5 (M1A1M1M1) gained then **SC M1A1** for any valid method of deducing upper bound, and similarly for lower bound	
			But upper bound attained $\Rightarrow \sin^2\theta = 0$ or 1 $\Rightarrow \sin 2\theta = 0$	M1	Dep on showing valid method for UB<=3	Or independent proof that not equal to 3
			So $\sin 2\theta \neq 0 \Rightarrow -1 \leq \frac{\sin 6\theta}{\sin 2\theta} < 3$	A1 [7]	full convincing overall argument	

Question			Answer	Marks	Guidance	
8	(i)		$ba = a^n \Rightarrow b = a^{n-1}$ But these are distinct elements so $ba \neq a^n$	M1 A1 [2]		
	(ii)		$ba = a^2b$ $\Rightarrow a^2ba = a^4b$ $\Rightarrow a^2ba = b$ $\Rightarrow a^2ba^4 = ba^3$ $\Rightarrow a^2b = ba^3$ $\Rightarrow ba = ba^3$ $\Rightarrow e = a^2$	M1 M1	or $b = a^2ba^3$, $a = ba^2b$, $a^2 = bab$ or $b = ba^2$ validly reach any equality which gives 2 distinct elements of the group as equal	
			Which is false, hence $ba \neq a^2b$	A1	Complete argument	
			If $ba = ab$ then (all element pairs would have to be commutative and so) G would be abelian.	M1	Do not award for G non-abelian $\Rightarrow ba \neq ab$	
			If $ba = b$ then $a = e$ so $ba \neq b$.	M1		
			So, by elimination of other possibilities, $ba = a^3b$	A1 [6]	Dependent on all previous marks	
	(iii)		$ba^2 = baa = a^3ba$ $= a^3a^3b = a^2b$	M1 A1 [2]	Use previous result Complete argument	

Question			Answer	Marks	Guidance	
	(iv)		<table><tr><td></td><td>e</td><td>a</td><td>a^2</td><td>a^3</td></tr><tr><td>e</td><td>e</td><td>a</td><td>a^2</td><td>a^3</td></tr><tr><td>a</td><td>a</td><td>a^2</td><td>a^3</td><td>e</td></tr><tr><td>a^2</td><td>a^2</td><td>a^3</td><td>e</td><td>a</td></tr><tr><td>a^3</td><td>a^3</td><td>e</td><td>a</td><td>a^2</td></tr></table>	B1	All correct	
			<table><tr><td></td><td>e</td><td>a^2</td><td>b</td><td>a^2b</td></tr><tr><td>e</td><td>e</td><td>a^2</td><td>b</td><td>a^2b</td></tr><tr><td>a^2</td><td>a^2</td><td>e</td><td>a^2b</td><td>b</td></tr><tr><td>b</td><td>b</td><td>a^2b</td><td>e</td><td>a^2</td></tr><tr><td>a^2b</td><td>a^2b</td><td>b</td><td>a^2</td><td>e</td></tr></table>	B1 M1 A1	Correct elements At least 12 out of 16 entries correct All correct	
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			Total	**72**		